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# Deformed free probability of Voiculescu

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Abstract. We introduce $r$-free product of states on the free product of $C^*$-algebras and r-free convolution of probability measures on real line. This makes unification of the free and Boolean probability. New classes of associative convolution of measures are considered related to Muraki-Lou examples.

The plan of this paper is following:

1. Introduction.
2. $r$-free product ($0 \leq r \leq 1$) of states.
    a. $r = 1$ – free product of Voiculescu
    b. $r = 0$ – Boolean product
3. $r$-Fock Space and $r$-Gaussian random variables.
4. $r$-free convolution of probability measures on $\mathbb{R}$.
5. Central limit theorem for $r$-convolution.
6. Remarks to Muraki-Lou convolution and $\Delta$-convolution of measures.

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## 1. Introduction

As each discrete group $G$ with $N$ generators is a homorphism image of the free group $\mathbb{F}_N$ in the same manner we would like to say that each "natural" probability is a deformation of the free probability of Voiculescu. In the papers [BS] [BKS] we considered deformed classical probability and we get so called $q$-deformed Fock space, $q$-second quantization and $q$-Gaussian processes. In this note we propose some versions of deformation of the free probability of Voiculescu using our technique coming from the conditional free product construction [BLS],[BW]. We use one parameter deformation $0 \le r \le 1$ and we get for $r = 1$ the free probability and for $r = 0$ the Boolean probability.

One of the main result of this paper is the construction on the free product of non-unital $C^*$-algebras $A_i$ with states $\varphi_i : A_i \to \mathbb{C}$, (we recall that by a state on a non-unital algebras we mean positive functional of norm 1), a new examples of states $\varphi : *A_i \to \mathbb{C}$ such that

i. $\varphi|_{A_i} = \varphi_i$

ii. (*Voiculescu property*) If $\varphi(a_i) = 0$ for $i = 1, \ldots, n$, and $a_j \in A_{i_j}, i_1 \neq i_2 \neq \ldots$,

then $\varphi(a_1 a_2 \ldots a_n) = 0$

In the case $r = 1$ we get the construction of the free product of states of Voiculescu. If r = 0, then we have the regular free product of states [B1,B2] (called also Boolean product). It has the property that if $a_j \in A_{i_j}, i_1 \neq i_2 \neq \ldots$, than $\varphi(a_1 a_2 \ldots a_n) = \varphi(a_1) \ldots \varphi(a_n)$.

Using the construction of $r$-free product of states ($0 \le r \le 1$) we can form the $r$-free convolution of probability measures on $\mathbb{R}$. Then we introduce the analogue of $R(\mu)$ – R-transform. The main ideas comes from the our paper [BLS,BW2].

As an example of application of R-transform we obtain central limit theorem for $r$-convolution. Our central limit measure $\mu_r$ is the "symmetrization" of the Marcenko-Pastur measure (the free Poisson measure) which Cauchy transform is of the form:

$$G_{\mu_r}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{r}{z - \cfrac{1}{z - \cfrac{r}{z - \ddots}}}}}$$

which is 2-periodic continued fraction and the measure $\mu_r$ is supported on two intervals if $0 \le r < 1$.

In the section 6 we propose some generalization of our construction so we can get some results of Muraki and Lou concerning *monotonic* convolution and then in central limit we have the arcsinus low that means the measure $\frac{1}{\pi}\sqrt{1-x^2}dx$.

## 2. *r*-Free Product of States

Let $A_i$ be a non-unital C$^*$-algebra $A_i$ with states $\varphi_i : A_i \to \mathbb{C}$. Let $\widetilde{A}_i$ be the unitalization of $A_i$ (i.e. $\widetilde{A}_i = A_i + \mathbb{C}\,1$) and we define the extension of $\varphi_i$ as $\widetilde{\varphi}_i(1) = 1, \widetilde{\varphi}_i|_{A_i} = \varphi_i$. Moreover let define a new state $\psi_i = r\varphi_i + (1-r)\delta_1$ where $\delta_1$ is the functional defined as

$$\delta_1(x) = \begin{cases} 0 & \text{if } x \neq \lambda 1 \\ \lambda & \text{if } x = \lambda 1 \end{cases}$$

then $\psi_i$ is also a state on unital algebra $\widetilde{A}_i$ and we can form the conditional free product state $\widetilde{\varphi}$ on the free product C$^*$-algebra $\widetilde{A} = *\widetilde{A}_i = \widetilde{*A_i}$:

$$\widetilde{\varphi} = *(\widetilde{\varphi}_i, \psi_i).$$

By [BLS] we knew that $\widetilde{\varphi}$ is a state on C$^*$-algebra $\widetilde{A}$. Hence also we get state $\varphi = \widetilde{\varphi}|_A$ on the free product of non-unital algebra $A = *A_i$. We call $\varphi = *_r\varphi_i$ - the *r-free product state*. From the construction of $\varphi$ we have the following properties:

(i) $\varphi|_{A_i} = \varphi_i$

(ii) if $a_j \in A_{i_j}, i_1 \neq i_2 \neq \ldots$, then

$$\widetilde{\varphi}[(a_1 - r\varphi(a_1)1\,(a_2 - r\varphi(a_2)1\ldots(a_n - r\varphi(a_n)1] = (1-r)^n\,\varphi(a_1)\ldots\varphi(a_n)$$

The formula (ii) is equivalent to:

$$\text{(iii)}\quad \begin{aligned} \varphi(a_1a_2\ldots a_n) &= r\sum_j \varphi(a_j)\varphi(a_1\ldots\overset{\vee}{a_j}\ldots a_n) - r^2\sum_{i<j}\varphi(a_i)\varphi(a_j)\varphi(a_1\ldots\overset{\vee}{a_i}\ldots\overset{\vee}{a_j}\ldots a_n) \\ &+\ldots+\left[(-1)^{n+1}r^n + (1-r)^n\right]\varphi(a_1)\ldots\varphi(a_n). \end{aligned}$$

We see that in the case $r = 0$ we get the regular free product of states (or Boolean). i.e. $\varphi(a_1a_2\ldots a_n) = \varphi(a_1)\ldots\varphi(a_n)$, if $a_j \in A_{i_j}, i_1 \neq i_2 \neq \ldots$. The class of such as states we founded in our paper from 1986 [B1,B2] which is a generalization of Haagerup states on the free product of group. [Haa1].

The most natural state on the group algebra of the free group $\mathbf{F}_N$ with the free generators $x_1, x_2, \ldots, x_N$ is the Haagerup state

$$H_q(g) = q^{l(g)},\ g \in \mathbf{F}_N$$

if $g = x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_k}^{n_k}, g \neq e, i_1 \neq i_2 \neq i_3 \neq \ldots, n_j \in \mathbb{Z}, l(g) = \sum_j |n_j|, l(e) = 0$. Since the full C$^*$-algebra C$^*$($\mathbb{F}_N$)=$\prod_{i=1}^{N} C^*(\mathbb{Z})^{(i)}$, where the product is the free product of C$^*$ algebras and

$$H_q = P_q *_0 \ldots *_0 P_q$$

is the Boolean free product, where $P_q(n) = q^{|n|}, (n \in \mathbb{Z})$ is the classical Fourier transform of the Poisson kernel.

One can see that in the case $r = 1$ our construction give Voiculescu free product of states in the case when the algebras $A_i$ are unital.

**Remark 2.1.** If $(A, \varphi) = *_r(A_i, \varphi_i)$ is the *r-free product* as defined above then if $a_i$ are in different algebras $A_i$, then $\varphi(a_1 a_2 \ldots a_n) = \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n)$.

Moreover if $a_1, a_2 \in A_i, b \in A_j, i \neq j$, then

(2.1) $$\varphi(a_1 b a_2) = r\varphi(a_1 a_2)\varphi(b) + (1-r)\varphi(a_1)\varphi(b)\varphi(a_2)$$

**Remark 2.2.** From the formula (2.1) we can infer that for $r \neq 0,1$ our $r$-free product is not associative i.e. if $(\varphi_1 *_r \varphi_2) *_r \varphi_3 = \varphi_1 *_r (\varphi_2 *_r \varphi_3)$ then $r = 0$ or $r = 1$. []

**Remark 2.3.** From the formula (iii) we see that the $r$-free product of states ($\varphi^{(r)} = *_r \varphi_i$) has *Voiculescu property*:

If $\varphi(a_i) = 0$ for all $j$ and $a_j \in A_{i_j}, i_1 \neq i_2 \neq \ldots$, then $\varphi(a_1 a_2 \ldots a_n) = 0$

Also for $r \neq 1$ $\varphi^{(r)}$ is different from the free product of Voiculescu.

**Problem 1.** Find other examples of states $F$ on $*(A_i, \varphi_i)$ such that:

(i) $F|_{A_i} = \varphi_i$

(ii) F satisfies Voiculescu property

**Problem 2.** If $r$-free product of states is again a state for $r > 1$?

2. $r$-Fock space and r-Gaussian random variables

Let $H$ be a real Hilbert space and $H_{\mathbb{C}}$ will be its complexification. We define the free Fock space $F(H_{\mathbb{C}}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H_{\mathbb{C}}^{\otimes n}$. Now we make deformation of the scalar product as follows:

**For** $x_n, y_n \in H_{\mathbb{C}}^{\otimes n}$ **we put**

$\langle x_n, y_n \rangle_r = r^k \langle x_n, y_n \rangle$ if $n = 2k$ or $n = 2k+1$, $k = 0, 1, 2, 3, \ldots$.

Moreover $\langle \Omega, \Omega \rangle_r = \langle \Omega, \Omega \rangle = 1$.

We can see $\langle x, x \rangle_r = \langle x, x \rangle$ for $x \in \mathcal{H}$. The completion of $\mathcal{F}(\mathcal{H}_{\mathbb{C}})$ with respect the scalar product $\langle , \rangle_r$ we called *r-Fock space* and will be denoted $\mathcal{F}(\mathcal{H}, r)$. Moreover for $f \in \mathcal{H}$ we define the $r$-creation operation $A^+(f)x_1 \otimes \ldots \otimes x_n = f \otimes x_1 \otimes \ldots \otimes x_n$ and the $r$-annihilation operator $A(f)$ such that $A(f)\Omega = 0$ and $A(f)x_1 \otimes \ldots \otimes x_n = \lambda_n \langle f, x_1 \rangle x_2 \otimes \ldots \otimes x_n$,

$$\lambda_n = \begin{cases} 1 & \text{if } n = 2k+1 \\ r & \text{if } n = 2k \end{cases}$$

**Proposition 3.1.**

(i) $A(f)^* = A^+(f), f \in \mathcal{H}$

(ii) $\|A(f)\| = \|A^+(g)\| = \max(1, r)\|g\|$

(iii) $A(f)A^+(g) = \lambda(N)\langle f, g\rangle$ where $\lambda(N)x_1 \otimes \ldots x_n = \lambda_n x_1 \otimes \ldots x_n$.

(iv) If $P$ is the orthogonal projection of $\mathcal{F}(\mathcal{H}, r)$ onto $\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes 2n}$, then $\lambda(N) = rP + (I - P) = I + (r-1)P$.

(v) If $A_i = A(e_i)$, where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$, then $\left\|\sum a_i \otimes A_i\right\|^2 = \max(1, r)\left\|\sum a_i a_i^*\right\|$.

Proof of (i) to (iii) follows directly from the definition. To get (v) let us observe that $\left\|\sum a_i \otimes A_i\right\|^2 = \left\|\left(\sum a_i \otimes A_i\right)\left(\sum a_j^* \otimes A_j^+\right)\right\| = \left\|\sum a_i a_j^* \otimes \lambda(N)\delta_{ij}\right\| = \left\|\sum a_i a_i^* \otimes \lambda(N)\right\| =$
$= \left\|\sum a_i a_i^*\right\|\left\|\lambda(N)\right\|$.

Since $\lambda(N)$ is the diagonal operator, therefore $\|\lambda(N)\| = \max(1, r)$.
Now we define r-*Gaussian* random variables. For f $\in \mathcal{H}$ $G(f) = A(f) + A^+(f)$ and for a bounded operator $T$ on $\mathcal{F}(\mathcal{H}, r)$ we define the vacuum state $\varepsilon(T) = \langle T\Omega, \Omega\rangle$.

**Corollary 3.2.**

$$\max\left\{\left\|\sum a_i a_i^*\right\|, \left\|\sum a_i^* a_i\right\|\right\} \le \left\|\sum a_i \otimes G_i\right\| \le 2\max(r,1)\max\left\{\left\|\sum a_i a_i^*\right\|, \left\|\sum a_i^* a_i\right\|\right\}.$$

We can now state the generalization of classical Wick formula (see []).

Let us recall that $NC_2(1, 2n)$ denote the set of all non-crossing 2-partitions on {1, 2, …, 2n},
$e(V) = \#\,\{B_j \in V : d_V(B_j)$ is even number}. Here $d_V(B_j)$ is the depth of the block $B_j$ in the partition $V$ as was defined in [].

**Theorem 3.3**. If $f_j \in \mathcal{H}$ then

$$(3.1)\qquad \varepsilon(G(f_1)G(f_2)\ldots G(f_{2n})) = \sum_{V \in NC_2(1,\ldots,2n)} \langle f_{i_1}, f_{j_1}\rangle \ldots \langle f_{i_n}, f_{j_n}\rangle r^{e(V)}.$$

The proof of the formula (3.1) follows from general result which was proven by us in the paper with Accardi [AB].

**Remark 3.4.** In the case $r = 1$ (the free Gaussian random variable) this formula wa obtained by R. Speicher,[Sp1].

If $r = 0$ (the Boolean Gaussian random variable) we have the following simple formula:

$$\varepsilon(G(f_1)G(f_2)\dots G(f_{2n})) = \langle f_1, f_2\rangle\langle f_3, f_4\rangle\dots\langle f_{2n-1}, f_{2n}\rangle.$$

**In the special case when $f_i = f$ we have**

$$\varepsilon(G(f)^k) = \begin{cases} \|f\|^{2n} & \text{if } k = 2n \\ 0 & \text{if } k = 2n+1. \end{cases}$$

Hence if $\|f\| = 1$, we see that the distribution of the Boolean Gaussian random variables $G(f)$ in the vacuum state $\varepsilon$ is the Bernoulli law $\mu_0 = \frac{1}{2}(\delta_1 + \delta_{-1})$.

Later on we will calculate the distribution of the $r$-free Gaussian random variables. Moreover in the Boolean case we have much more that corollary 3.2.

**Corollary 3.5.**

$$\left\|\sum a_i \otimes G_i\right\| = \max\left\{\left\|\sum a_i a_i^*\right\|, \left\|\sum a_i^* a_i\right\|\right\} \tag{3.2}$$

The proof of (3.2) follows from the following observation for the block matrices:

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} = \begin{pmatrix} TT^* & 0 \\ 0 & T^*T \end{pmatrix}$$

and $T = \sum a_i \otimes G_i = \left(\begin{array}{c|ccc} 0 & a_1 & \cdots & a_n \\ \hline a_1^* & & & \\ \vdots & & 0 & \\ a_n^* & & & \end{array}\right)$

**Problem 3**. Let $VN_r(N) = VN_r(G_1, \dots, G_N)$ will be the von Neumann algebra generated by $G_1, \dots, G_n$ in the $r$-Fock space $\mathcal{F}(\mathcal{H}, r)$.

If $r = 0$, then $VN_0(N) = M_N(\mathbb{C})$.
If $r = 1$, then $VN_1(N)$ is the free group factor – VN($\mathbb{F}_N$).
Try to verify if $VN_r(N)$ is also a factorial von Neumann algebra for $0 < r < 1$.
When does exist a trace on $VN_r(N)$?

### 3. $r$-Free Convolution of Probability Measures on $\mathbb{R}$.

In this section we will work mainly with probability measures $\mu$ on $\mathbb{R}$ with compact support ($\mu \in \mathcal{P}^c$). Let

$$m_k(\mu) = \int_{\mathbb{R}} x^k d\mu(x), \ \mathrm{k} = 0, 1, 2, \dots$$

and we treat the measure $\mu$ as a state on the algebra of polynomials $\mathbb{C}\langle X\rangle : \mu\left[X^k\right] = \mu_k(\mu)$.
If we take two probability measures $\mu_1, \mu_2 \in \mathcal{P}^c$ we define their
*r-free convolution* $(\mu_1 \circledR \mu_2)$ as follows:

(4.1) $$(\mu_1 \circledR \mu_2)[X^k] = (\mu_1 *_r \mu_2)\left[(X_1 + X_2)^k\right], k = 0, 1, \ldots,$$

here $(\mu_1 *_r \mu_2)$ is the $r$-free product of states on the algebra of non-commutative polynomials $\mathbb{C}\langle X_1, X_2\rangle$.

On the other hand using our conditionally free product of pairs of probability measure as was done in [BLS]. The $r$-convolution of measure $\mu_1$, $\mu_2$ is the measure $\mu$ denoted as $\mu_1 \circledR \mu_2$ can be obtained in the following way:

$(\mu_1, V_r(\mu_1)) \boxplus (\mu_2, V_r(\mu_2)) = (\mu, \nu)$, where $\nu$ is the Voiculescu free product $V_r(\mu_1) \boxplus V_r(\mu_2) = \nu$.

Here $V_r(\mu) = r\mu + (1-r)\delta_0$.
This implies that

$$\int x^k dV_r(\mu)(x) = r\int x^k d(\mu)(x),\ k \geq 1$$

and therefore using the conditional $R$-transform $R_\mu(k) = R(\mu, V_r(\mu))(k)$ we have the following formula for calculation of moments for $\mu \in \mathcal{P}^c$:

**(4.2)** $$\int x^n d\mu(x) = \sum_{V \in NC(n)} R_\mu(V) r^{e(V)},$$

where $R_\mu(V) = \prod_{B \in V} R_\mu(\#B)$ and $e$ is a suitable function on the set of non crossing partitions $NC(2n)$.

The important property of the function $e$ is that $e(V_0) = 1$, where $V_0 = \{\{1, \ldots, n\}\}$.
The formula (4.2) is obtained directly from the formula (4.3) from the paper [BLS]

(4.3) $$m_n(\mu) = \sum_{k=1}^{n} \sum_{\substack{l(1),\ldots l(k)\geq 0\\ l(1)+\ldots l(k)=n-k}} R_\mu(k) m_{l(1)}(\mu) \ldots m_{l(k-1)}(\mu) m_{l(k)}(\mu) r^{n-k-l(k)}$$

The formula (4.2) implies that $r$-free convolution of probability measure is *associative*.
Moreover if $\delta_x$ is Dirac measure at point $x \in \mathbb{R}$, then $\delta_x \circledR \delta_y = \delta_{(x+y)}$.

**Problem 4**

From theorem (3.3) we know that for 2-non-crossing partition $V$,
$e(V) = \#\{B \in V : d_V(B) \text{ is even}\}$. Find a description of the function for <u>all</u> non-crossing partitions.

After this consideration we can now formulate our result:
**Proposition 4.1**

If $\mu \in \mathcal{P}^c$ and for $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ then

(4.4) $$\frac{1}{G_\mu(z)} = z - R_\mu(rG_\mu(z) + (1-r)\frac{1}{z}),$$ where

$$G_\mu(z)=\int\frac{d\mu(x)}{z-x},\; R_\mu^{(r)}(z)=R_\mu(z)=\sum_{k=1}^{\infty}R_\mu(k)z^k\,.$$

The proof of (4.4) is the reformulation of the corresponding formula from the theorem 5.2 in [] in the particular case where the measure $\nu=r_\mu+(1-r)\delta_0$.

Therefore $G_\nu(z)=rG_\mu(z)+(1-r)\frac{1}{z}$.

The details are left to the reader.

**Remark 4.2**

If $r=1$ the fact (4.4) is the Voiculescu theorem for the free cumulant. If $r=0$ then we have Boolean cumulant formula of Speicher and Wourudi[SW]:

$$\frac{1}{G_\mu(z)}=z-R_\mu^{(0)}\left(\frac{1}{z}\right)$$

## 4. Central Limit Theorem

This section is devoted to the main result of this paper.

**Theorem 5.1**

Let $X_i=X_i^*\in(A,\varphi)$, where A is a $C^*$ algebra with a state $\varphi$ and $X_1,X_2,\ldots$ are *r-free random variables* in the probabilistic system $(A,\varphi)$. That means that $A=*_rA_i$, $\varphi=*_r\varphi_i$ and $X_i=X_i^*\in(A_i,\varphi_i)$. Assume that:

(i) $\varphi(X_i)=0$

(ii) $\varphi(X_i^2)=1$

(iii) $\|X_i\|<C$.

If we take $S_N=\frac{1}{\sqrt{N}}\sum_1^N X_i$, then $\lim_{N\to\infty}\varphi(S_N^k)=\int x^k d\mu_r(x)$, where the probability measure $\mu_r=\frac{1}{2}\left(f_r(x)\chi_{I_r}+f_r(-x)\chi_{(-I_r)}\right)dx$ and $f_r(x)=\frac{1}{\pi x}\sqrt{4r-(x^2-(1+r))^2}$.

Moreover the Cauchy transform of the measure $\mu_r$ has the following continued fraction form:

$$G_{\mu_r}(z)=\cfrac{1}{z-\cfrac{1}{z-\cfrac{r}{z-\cfrac{1}{z-\cfrac{r}{z-\ddots}}}}}$$

Proof. The limit measure $\mu_r$ is such that

(5.1) $$R^{(r)}_{\mu_r}(z) = z.$$

The argument is almost the same as in the proof of the free probability central limit theorem so we omit it (see [VDN]).

Hence the Cauchy transform $G(z) = G_{\mu_r}(z)$ of measure $\mu_r$ satisfies the equation:

(5.2) $$\frac{1}{G(z)} = z - \left( rG(z) + (1-r)\frac{1}{z} \right).$$

Now we will show that $G(z) = H(z)$, where $H(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{r}{z - \cfrac{1}{z - \cfrac{r}{z - \ddots}}}}}$.

First we see that $H(z) = z - \dfrac{1}{z - rH(z)}$. So we see that $H = H(z)$ satisfies the following equation:

(5.3) $$zrH^2 + (1 - z^2 - r)H + z = 0.$$

But from the formula (5.2) follows that $G(z)$ is also the root of the equation (5.3). Therefore $G(z) = H(z)$.

Now we want to calculate explicit form of the limit measure $\mu_r$. For this let us observe that the following fact holds:

(5.4) $$m_{2n}(\mu_r) = \frac{1}{r} m_n(p_r),$$

where $n > 0$ and $p_r$ is the free Poisson measure with intensivity $r$.

To show (5.4) let us recall that

(5.5) $$m_n(p_r) = \sum_{V \in NC(n)} r^{\#(V)}$$

For the proof of that fact see [Sp1,BLS]. On the other side let us calculate the moments of our limit measure $\mu_r$ and the free Poisson measure $p_r$ using our theorem from [AB]:

$$m_2(\mu_r) = 1 \qquad m_1(p_r) = r$$
$$m_4(\mu_r) = 1 + r \qquad m_2(p_r) = r(1+r)$$
$$m_6(\mu_r) = 1 + 3r + r^2 \qquad m_6(p_r) = r(1 + 3r + r^2)$$

**and by the induction argument we get the proof of (5.4).**

Hence by small calculation we obtain the equation:

(5.6) $$G_{\mu r}(z) = \frac{1}{r} z G_{p_r}(z^2) + (1 - \frac{1}{r})\frac{1}{z}.$$

Now in the proof we need the following simple lemma:

Lemma 5.2 Let $f \in L^1(\mathbb{R})$, and supp($f$)=I $\subset \mathbb{R}^+$ and

$$\tilde{f}(x) = \frac{1}{2}\left(f(x)\chi_I(x) + f(-x)\chi_{(-I)}(x)\right)$$

then the Cauchy transform of $\tilde{f}$ is of the form:

(5.7) $$G_{\tilde{f}}(z) = zG_F(z^2), \text{ where } F(x) = \frac{f(\sqrt{x})}{2\sqrt{x}}.$$

Since $P_r = (1-r)\delta_0 + F_r(x)\,dx$, where $0 \le r \le 1$.

This implies that:

(5.8) $$G_{\mu_r}(z) = zG_{F_r}(z^2) = G_{\tilde{f}_r}(z).$$

Therefore by lemma 5.2 we get that

(5.9) $$\mu_r = \tilde{f}_r(x)\,dx = \frac{1}{2}\left(f(x)\chi_I(x) + f(-x)\chi_{(-I)}(x)\right)dx.$$

As we knew (see [VDN,BLS])

$$F_r(x) = \frac{1}{2\pi x}\sqrt{4r - (x-(1+r))^2}.$$

Since $f_r(x) = 2xF_r(x^2) = \dfrac{1}{\pi x}\sqrt{4r - (x^2-(1+r))^2}$

and supp $f_r = I_r$, where $I_r$ is interval of the form $I_r = \left[1-\sqrt{r}, 1+\sqrt{r}\right]$, therefore this completes the proof of theorem 5.1.

**Remark 5.3** If $r = 1$, we have $f_1(x) = \dfrac{1}{\pi}\sqrt{4-x^2}$.

Therefore $\mu_1 = \dfrac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}\,dx$ - so this is semicircle low of Wigner (free Gaussian random variables).

**Remark 5.4** It is also possible to calculate the measure $\mu_r$ for $r > 1$ and then we can see that measure has a one atom at 0 (see [K]). It will be interesting to see why that measure is connected with quasi-free free state considered by Shlyakhtenko [Sh]?

## 5. Remarks on Muraki-Lou convolution and $\Delta$-convolution.

In this chapter we present some generalization of $r$-free convolution of probability measures. For this let $C: \text{Prob}(\mathbb{R}) \to \text{Prob}(\mathbb{R})$ will be some map and $\text{Prob}(\mathbb{R})$ is the set of all probability measures on the real line. We will define $C$-free convolution of measures as follows:

(6.1) $$(\mu_1, C(\mu_1)) \boxplus (\mu_2, C(\mu_2)) = (\mu, C(\mu_1) \boxplus C(\mu_2)),$$

where the convolution of the pairs of measure is *conditionally free convolution* ([BLS]).

The formula (6.1) defines $C$-free convolution of $\mu_1 \copyright \mu_2 = \mu$.

In the special case when $C(\mu) = V_r(\mu) = r\mu + (1-r)\delta_0 = (r\delta_1 + (1-r)\delta_0) \Box \mu$ by above method we obtain again $r$-free convolution. Here $\Box$ denote the multiplicative convolution of probability measures on real line.

Another example of deformed free convolution was presented by Wysoczanski and myself (see [BW1,BW2]). This corresponds to $C$-convolution, where $C = U_t$ ($t \geq 0$) is defined by the equation $\frac{1}{G_{\mu(t)}(z)} = \frac{t}{G_\mu(z)} + (1-t)z$, where $\mu(t) = U_t(\mu) = C(\mu)$. In that example the central limit measure $K_t$ is the Kesten measure which is the spectral measure for the random walks on the free group $\mathbf{F}_N$ and the parameter $t = 1 - \frac{1}{2N}$. The Cauchy transform of the measure $K_t$ has following continued fraction form:

$$G_{K_t}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{z - \cfrac{t}{z - \ddots}}}}}$$

The rest of that chapter will be devoted to the special class of convolution – called $\Delta$-*convolution* which corresponds to the map $C: \mathrm{Prob}(\mathbb{R}) \to \mathrm{Prob}(\mathbb{R})$ done by the multiplicative convolution $\Box$ on the real line by the suitable measure $\omega$; so we define $C(\mu) = \mu \Box \omega$ or in another words if $\delta_n = \int x^n d\omega(x)$, then $m_n(C(\mu)) = \delta_n\, m_n(\mu)$, $n = 0, 1, \ldots$. In that case our $\Delta$-convolution is associative, since we have $R$-transform – $R_\mu^\Delta = R^\Delta(\mu)$ which make linearization of our $C$-convolution. That exactly means that $R^\Delta(\mu_1 © \mu_2) = R^\Delta(\mu_1) + R^\Delta(\mu_2)$. Also there is a nice connection between $R^\Delta$-cumulants and moments done by formula:

$$\int_{\mathbb{R}} x^n d\mu(x) = \sum_{V \in NC(n)} R_\mu^\Delta(V)\, t(V, \Delta)$$

for proper function $t(\cdot, \Delta)$ on non-crossing partition set $NC(n)$. Now we can present a generalization of central limit theorem for $\Delta$-convolution.

We recall that dilatation $D_s$ of the measure $\mu$ is defined as $D_s(\mu)(E) = \mu(\frac{1}{s}E)$ for Borel set $E \subset \mathbb{R}$ and $s \neq 0$.

**Theorem 6.1**

Let $\mu_i \in \mathrm{Prob}(\mathbb{R})$ and all moments of measures $\mu_i$ are finite. Assume that

(i) $\int x d\mu_j(x) = 0$

(ii) $\int x^2 d\mu_j(x) = 1$

(iii) $\left|\int x^k d\mu_j(x)\right| \leq B_k$, for all $j$, then the measures $S_N = D_{\frac{1}{\sqrt{N}}}(\mu_1)\copyright\ldots\copyright D_{\frac{1}{\sqrt{N}}}(\mu_N)$ weakly tends to limit measure $\mu$.

$D_s(\mu)(E) = \mu(\frac{1}{s}E)$ for Borel set $E \subset \mathbb{R}$.

Moreover

$$G_\mu(z) = \frac{1}{z - G_{C(\mu)}(z)} \quad \left(\Leftrightarrow R_\mu^\Delta(z) = z\right). \tag{6.2}$$

The proof is the same like theorem 5.1 so we omit it.

**Corollary 6.2**

If we take as a measure $d\omega(x) = |x|\chi_{[-1,1]}\,dx$ then the corresponding $\Delta$-convolution is related to the convolution discovered by Muraki-Lou and the central limit measure is the arcsinus low $\frac{1}{\pi}\frac{1}{\sqrt{2-x^2}}dx$.

In the proof of the corollary use the fact that $C(\mu) = \frac{1}{\pi}\sqrt{2-x^2}\,\chi_{[-\sqrt{2},\sqrt{2}]}\,dx$ if $\mu = \frac{1}{\pi\sqrt{2-x^2}}dx$. Moreover

$$G_\mu(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{1/2}{z - \cfrac{1/2}{z - \cfrac{1/2}{z - \ddots}}}}}$$

and

$$G_{C(\mu)}(z) = \cfrac{1}{z - \cfrac{1/2}{z - \cfrac{1/2}{z - \cfrac{1/2}{z - \cfrac{1/2}{z - \ddots}}}}}$$

so evidently the equation (6.2) is satisfied.

**Problem 5**

Characterize all central limit measures for all moment sequences $\Delta = (\delta_n)$ in the case of $\Delta$-convolution.

**Remark 6.3**

One can show that the classical Gauss measure $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$ is not the central limit measure for any $\Delta$-convolution. Hint: Use the equation (6.2).

**Remark 6.4**

In our case $\Delta$-convolution are commutative but as was shown by Muraki his monotonic convolution is not comutative.

**Remark 6.5**

If we take $C(\mu) = \mu \,\square\, \delta_s = D_s(\mu)$ i.e. that $\delta_n = m_n(\omega) = s^n$ then we obtain a quite interesting deformation of the free convolution. The case $s = 0$ is again Boolean convolution. The corresponding central limit measure for that convolution is the measure $\mu = \mu_s$, for which Cauchy transform satisfies the equation

$$G_\mu(z) = \frac{1}{z - \frac{1}{s} G_\mu\left(\frac{z}{s}\right)}.$$

If $0 \le s < 1$, then one can verify that our central limit measure $\mu_s$ is discrete measure and the orthogonal polynomials with respect that measure satisfy equation:

$$xP_n(x) = P_{n+1}(x) + s^{2n-2} P_{n-1}(x), \quad P_0(x) = 1, \quad P_1(x) = x.$$

That last fact about orthogonal polynomials is equivalent (by Stieltjes theorem) that the Cauchy transform of the measure $\mu_s$ has the form:

$$G_{\mu_s}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{s^2}{z - \cfrac{s^4}{z - \cfrac{s^6}{z - \ddots}}}}}$$

We will finish our note with the following problem related to the last example of convolution.

**Problem 6**

(i) Calculate the support of the measure $\mu_s$ for $s < 1$.

(ii) Consider also the case $s > 1$.

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